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# Fault-Tolerant Cycle Embedding in Dual-Cube with Node Faulty

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**Abstract:** *A low-degree dual-cube was proposed as an alternative to the hypercubes. A dual-cube DC(m) has* $m+1$ *links per node where m is the degree of a cluster (m-cube) and one more link is used for connecting to a node in another cluster. There are* $2^{m+1}$ *clusters and hence the total number of nodes is* $2^{2m+1}$ *in a DC(m). In this paper, by using Gray code, we show that there exists a faulty-free cycle containing at least* $2^{2m+1}-2f$ *nodes with* $f \le m-1$ *faulty nodes.*

**Keywords:** Interconnection networks, hypercube, hamiltonian cycle, Gray code, fault-tolerant embedding

## 1 Introduction

The binary hypercube has been widely used as the interconnection network in a wide variety of parallel systems such as Intel iPSC, the nCUBE [4], the Connection Machine CM-2 [15], and SGI Origin 2000 [12]. A hypercube network of dimension $n$, or $n$-cube, contains up to $2^n$ nodes and has $n$ edges per node. If unique $n$-bit binary addresses are assigned to the nodes of the hypercube, then an edge connects two nodes if and only if their binary addresses differ in a single bit. Because of its elegant topological properties and the ability to emulate a wide variety of other frequently used networks, the hypercube has been one of the most popular interconnection networks for parallel computer/communication systems.

However, the conventional hypercube has a major shortage, that is, the number of edges per node in a system increases logarithmically as the total number of nodes in the system increases. Since the number of links is limited to eight per node with current IC technology, the total number of nodes in a hypercube parallel computer is restricted to several hundreds. Therefore, it is interesting to develop an interconnection network which keeps most of topological properties of the hypercube, and has more nodes in the system than the hypercube with the same number of edges per node.

Several variations of the hypercube have been proposed in the literature. Some variations focused on the reduction of diameter of the hypercube, such as folded hypercube [1] and crossed cube [2]; some focused on the reduction of the number of edges of the hypercube, such as cube-connected cycles [10] and reduced hypercube [17]; and some focused on the both, like hierarchical cubic network [3]. Generally, the variations of the hypercube that reduce the diameter, e.g. crossed cube and hierarchical cubic network, will not satisfy the following key property in the hypercube: each node can be represented by a unique binary number such that two nodes are connected by an edge only if the two binary numbers differ in one bit. This key property is at the core of many algorithmic designs for efficient routing and communication.

A new interconnection network for large parallel systems called *dual-cube* (DC) has been introduced recently [7] [8]. The dual-cube shares the desired properties of the hypercube (e.g., the key property of the hypercube mentioned above), and increases tremendously the total number of nodes in the system compared with the hypercube of the same node degree. The size of the dual-cube can be as large as thirty thousands with up to eight links per node. It is practically important to refine the hypercube networks such that the size of the network can be increased while the number of the links per node is limited by the technology.

A *hamiltonian cycle* of an undirected graph $G$ is a simple cycle that contains every node in $G$ exactly once. A *hamiltonian path* in a graph is a simple path that visits every node exactly once. A hamiltonian path can be obtained from a hamiltonian cycle by removing any one link from that cycle. A graph that contains a hamiltonian cycle is said to be *hamiltonian*. $G$ is *k-link hamiltonian* if it remains hamiltonian after removing any $k$ links. It is clear that if graph $G$ is $k$-connected then $G$ can be at most $(k-2)$-link hamiltonian.

Constructing fault-free cycle is important for linear array or ring embedding. Previous results about fault tolerant cycle embedding in networks are as follows. The $n$-cube is $(n-2)$-link hamiltonian [6]. The $n$-dimensional folded hypercube is $(n-1)$-link hamiltonian [16]. The $n$-dimensional star graph is $(n-3)$-link hamiltonian [14]. A $k$-ary undirected de Bruijn graph is $(k-1)$-link hamiltonian [11]. An $(m+1)$-connected DC($m$) is $(m-1)$-link hamiltonian [9].

The problem of faulty-node tolerant cycle embedding is to find a cyle in a network with some faulty nodes. The cycle length depend on the number of faulty nodes. For example, an $n$-cube with $f$ faulty nodes can embed a fault-free cycle cantaining at least $2n-2f$ nodes, where $f \leq n-1$ [13]. An $n$-dimensional star graph with $f$ faulty nodes can embed a fault-free cycle cantaining at least $n!-2f$ nodes, where $f \leq n-3$ [5]. An $d$-ary $n$-dimensional undirected de Bruijn graph with $f$ faulty nodes can embed a fault-free cycle cantaining at least $d^n-nf-1$ nodes, where $f \leq d-1$ [11]. In this paper, we show that a DC($m$) with $f$ faulty nodes can embed a fault-free cycle cantaining at least $2^{2m+1}-2f$ nodes, where $f \leq m-1$.

The rest of this paper is organized as follows. Section 2 describes the dual-cube architecture. Section 3 constructs a hamiltonian cycle in a DC($m$). Section 4 shows that there exists a faulty-free cycle containing at least $2^{2m+1}-2f$ nodes in a DC($m$) with $f \leq m-1$. Section 5 concludes the paper and presents some future research directions.

## 2 Dual-cube Architecture

A dual-cube uses hypercubes as basic components. Each hypercube component is referred to as a *cluster*. Assume that the number of nodes in a cluster is $2^m$. In a dual-cube, there are two *classes* with each class consisting of $2^m$ clusters. The total number of nodes is $2^m \times 2^m \times 2$, or $2^{2m+1}$. Each node in a dual-cube has $m+1$ links: $m$ links are used within cluster to construct an $m$-cube and a single link is used to connect a node in a cluster of the other class. There is no link between the clusters of the same class. If two nodes are in one cluster, or in two clusters of distinct classes, the distance between the two nodes is equal to its *Hamming distance* (the number of bits where the addresses of the two nodes have different values). Otherwise, it is equal to the Hamming distance plus two: one for entering a cluster of the other class and one for leaving.

An $(m+1)$-connected dual-cube DC($m$) is an undirected graph on the node set $\{0,1\}^{2m+1}$ and there is an edge between two nodes $u = (u_{2m} \ldots u_0)$ and $v = (v_{2m} \ldots v_0)$ if and only if the following conditions are satisfied:

1. $u$ and $v$ differ exactly in one bit position $i$,
2. if $0 \leq i \leq m-1$ then $u_{2m} = v_{2m} = 0$ and
3. if $m \leq i \leq 2m-1$ then $u_{2m} = v_{2m} = 1$.

Intuitively, the set of nodes $u$ of form $(0u_{2m-1} \ldots u_m * \ldots *)$, where $*$ means "don't care", constitutes an $m$-dimensional hypercube. We call these hypercubes clusters of class 0. Similarly, the set of nodes $u$ of form $(1 * \ldots * u_{m-1} \ldots u_0)$ constitutes an $m$-dimensional hypercube and we call them clusters of class 1. The edge connecting two nodes in two clusters of distinct classes is called *cross-edge*. In the other word, $e = (u : v)$ is a cross-edge if and only if $u$ and $v$ differ in the leftmost bit.

Each node in a DC($m$) is identified by a unique $(2m+1)$-bit number, an *id*. Each *id* contains three parts: *class_id*, *cluster_id* and *node_id*. In the following discussion, we use $id = (class\_id, cluster\_id, node\_id)$ to denote the node address where *class_id* is a 1-bit number, *cluster_id* and *node_id* are $m$-bit numbers. The bit-position of *cluster_id* and *node_id* depends on the value of *class_id*. If $class\_id = 0$ ($class\_id = 1$), then *node_id* (*cluster_id*) is the rightmost $m$ bits and *cluster_id* (*node_id*) is the next (to the left) $m$ bits. The cluster containing node $u$ is denoted as $C_u$. For any two nodes $u$ and $v$ in a DC($m$), $C_u = C_v$ if and only if $u$ and $v$ are in the same cluster.

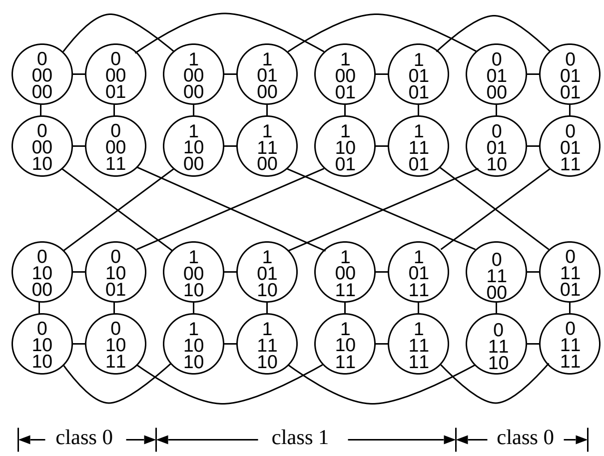



**Figure 1. A dual-cube DC(2)**

Figure 1 depicts a DC(2) network. In each node, *class_id* is shown at the top position. For the nodes of class 0 (class 1), *node_id* (*cluster_id*) is shown at the bottom and *cluster_id* (*node_id*) is shown at the middle. Figure 2 shows a DC(3). Notice that only those cross-edges connecting to cluster 0 of class 1 are shown, the other cross-edges are omitted for simplicity.

The dual-cube has a binary presentation of nodes, similar to a hypercube, in which two nodes are connected by an edge only if their addresses differ in one bit. This feature is the key for designing efficient routing and communication algorithms in dual-cube. Another important feature of a dual-cube is that, within the given bound to the number of links per node, say $m+1$, the network can have up to $2^{2m+1}$ nodes, more than the hypercube or the hierarchical cubic network can have.

The DC($m$) topological properties are given in [7] and

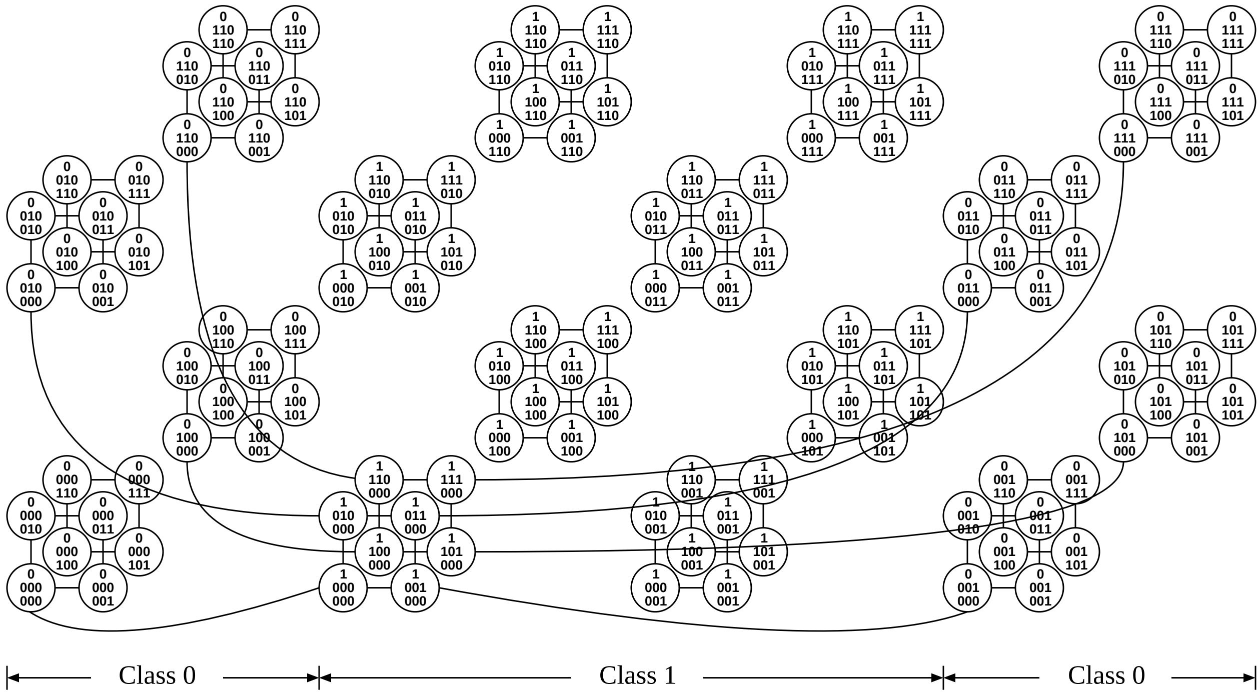



**Figure 2. A dual-cube DC(3)**

the collective communication schemes in DC($m$) can be found in [8].

## 3 Hamiltonian Cycle in Dual-Cube

In [9], it was proved that the dual-cube is $(m-1)$-link hamiltonian. That is, if a DC($m$) contains $m-1$ faulty links, there exists a cycle that cantains all the nodes. In this section, we show how to construct hamiltonian cycles in dual-cube because it is needed for fault tolerant cycle embedding in dual-cube with faulty nodes.

The key for constructing a hamiltonian cycle in a DC($m$) is to construct a *virtual hamiltonian cycle* that connects all $2^{m+1}$ clusters in DC($m$). The virtual hamiltonian cycle in a DC($m$) contains equal numbers of cube-edges and cross-edges; the cube-edges and the cross-edges are interleaved. To construct a fault-free hamiltonian cycle in a DC($m$) with up to $m-1$ faulty links, we need to put some constraints on the cube-edges in the virtual hamiltonian cycle since a hamiltonian path inside a cluster with faulty links might have fixed end nodes.

We use $0^{\{i\}}$ to denote a bit pattern $0\ldots0$ of $i$ bits. The hamiltonian cycle in an $n$-cube can be constructed by the *binary reflected Gray code*. A *Gray code* for binary numbers is a listing of all $n$-bit numbers so that successive numbers, including the first and last, differ in exactly one bit position. The best known example of the Gray codes is the *binary reflected Gray code* which can be described as follows. If $P(n)$ denotes the listing for $n$-bit numbers, then $P(1) = (0,1)$. For $n$ greater than 1, $P(n)$ is formed by taking the list for $P(n-1)$ with each number prefixed by 0 then following that list by the reverse of $P(n-1)$ with each number prefixed by 1. For example, $P(2) = (00,01,11,10)$, $P(3) = (000,001,011,010,110,111,101,100)$, and so on. Since the first and last numbers of $P(n)$ also differ in one bit position, the code is in fact a cycle. In an $n$-cube, there is a link connecting two nodes if their numbers differ in one bit position: connecting the adjacent nodes, also the first and last nodes, in the binary reflected Gray code list with links, a hamiltonian cycle is formed.

Let $D(n)$ denote the listing for the dimensions which changed in the number sequence in the reflected Gray code list. Then, $D(1) = 0$. For $n$ greater than 1, $D(n)$ is formed by taking the list $D(n-1)$ two times and inserting a number $n-1$ into between the two lists. For example, $D(2) = (0,1,0)$, $D(3) = (0,1,0,2,0,1,0)$, $D(4) = (0,1,0,2,0,1,0,3,0,1,0,2,0,1,0)$, and so on. That is, $D(n)$ can be constructed recursively as follows: $D(n) = (D(n-1),n-1,D(n-1))$ if $n > 1$, and $D(1) = (0)$. Note that reversing the node numbers performed in the generation of the reflected Gray code does not affect the dimensions which change in the sequence of the reflected Gray code: we just copy $D(n-1)$ to the second half part of $D(n)$. We call $D(n)$ a *reflected dimension list*.

In the what follows, we use $(u \rightarrow v)$ to denote a path or a cycle, and $(u : v)$ to denote a link connecting nodes $u$ and

$v$. Also, a format like (00 : 01 : 11 : 10) denotes a path or a cycle. The following algorithm is for generating a hamiltonian cycle $P$ in an $n$-cube with the reflected dimension list. The $\oplus$ does bit-wise exclusive OR operation.

```
Algorithm 1 (cubeHC(n))
begin            /* build a hamiltonian cycle P in an n-cube */
    D(n) = DL(n);        /* D(n): reflected dimension list */
    w = 0;                          /* starting from node 0 */
    P = w;                   /* P is the hamiltonian cycle */
    for each dimension number i in D(n) do
        w = w ⊕ 2^i;                  /* find the next node */
        P = P : w;                    /* add the node into P */
    endfor
end
Procedure DL(n)
begin /* build a reflected dimension list for an n-cube */
    if (n == 1) return (0);
    else return (DL(n−1), n−1, DL(n−1));
end
```

Note that the reflected Gray code or reflected dimension list is just one solution of the Gray codes. By renumbering the node numbers (exchanging bit positions of the all node numbers), we can have $n!$ different Gray code sequences. Furthermore, since there are $2^n$ links in the cycle, breaking a different link will get a different path: there are $2^n n!$ hamiltonian paths with different patterns in an $n$-cube.

Next, we add a condition to let a hamiltonian cycle contain a given link. This is needed for constructing a fault-free hamiltonian cycle in a dual-cube with faulty links.

**Lemma 1.** *Given any link $e = (u : v)$ in an $n$-cube where $u$ and $v$ are two distinct nodes and $d(u,v) = 1$, there is a hamiltonian cycle going through $e$.*

***Proof***: The lemma can be proved by renumbering every node in the cube with a mapping function $f(x)$ so that $u' = f(u) = 0^{\{n-1\}}0$ and $v' = f(v) = 0^{\{n-1\}}1$. Then a hamiltonian cycle is built by Algorithm 1 with the new numbers. Finally, the hamiltonian cycle denoted with the original node numbers is obtained by applying $f^{-1}(x)$ to every node number in the built cycle with the new numbers, where $f^{-1}(x)$ is the reverse of function $f(x)$: $u = f^{-1}(u')$ and $v = f^{-1}(v')$. One possible $f(x)$ does exclusive OR operation with $u$ on every node number so that node $u$ will have a new number $0^{\{n-1\}}0$, and then exchanges bit positions so that the node $v$ will have a new number $0^{\{n-1\}}1$. $\square$

By removing $e = (u : v)$ from the hamiltonian cycle constructed in Lemma 1, we get a hamiltonian path from node $u$ to node $v$, $(u \to v)$. We name the procedure that generates such a path as cubeHP$(m,u,v)$. This procedure will be used in constructing a hamiltonian cycle in a DC($m$).

A hamiltonian cycle in a DC($m$) can be constructed as follows. First, we can build a *virtual* hamiltonian cycle, $V(m)$, which connects all the clusters with only two neighboring nodes, $u$ and $v$ for instance, from each cluster (Figure 3). It is called *virtual* since the cube-edge $e = (u : v)$ in the cycle will be replaced with a hamiltonian path $(u \to v)$ in that cluster to form a "real" hamiltonian cycle in DC($m$).

The construction of the virtual hamiltonian cycle can be done by using an *extended double-dimension list*, or $EDD(m)$, defined as follows. Let the reflected double-dimension list be $DD(m) = (DD(m-1), m-1, m-1, DD(m-1))$ if $m > 1$, and $DD(1) = (0,0)$. Then the extended double-dimensional list $EDD(m) = (DD(m), m-1, m-1)$. Since there are two classes in a dual-cube, $EDD(m)$ doubles each dimension number in an extended list which consists of $D(m)$ plus the highest dimension $m-1$. For example, $EDD(2) = (0,0,1,1,0,0,1,1)$, $EDD(3) = (0,0,1,1,0,0,2,2,0,0,1,1,0,0,2,2)$, and so on. Then the virtual hamiltonian cycle can be constructed with $EDD(m)$. For example, $V(2) = ($00000, 00001, 10001, 10101, 00101, 00111, 10111, 11111, 01111, 01110, 11110, 11010, 01010, 01000, 11000, 10000$)$. Second, in each cluster we replace the edge $e = (u : v)$ with a hamiltonian path $(u \to v)$ to connect all the nodes in the cluster to form a hamiltonian cycle in DC($m$).

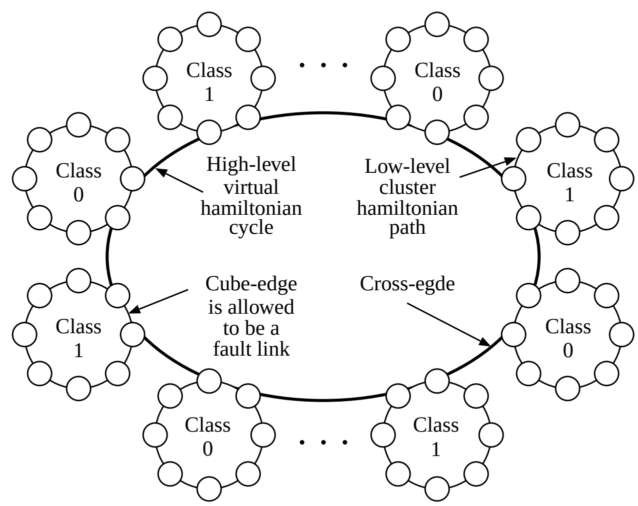



**Figure 3. Virtual hamiltonian cycle**

This virtual hamiltonian cycle could be considered as a high-level cycle which connects all the clusters. Note that only two neighboring nodes in each cluster are contained in the virtual hamiltonian cycle, and cube-edges and cross-edges are interleaved. Because there are two classes in a DC($m$) and each class has $2^m$ clusters, the virtual hamiltonian cycle contains $2^m \times 2 \times 2$, or $2^m \times 4$ nodes. If we group 4 nodes, whose right $m$ bits of the addresses are the same (e.g., 00001, 10001, 10101, 00101), into a *big* node, the virtual hamiltonian cycle contains $2^m$ big nodes. Therefore, the algorithm to construct the virtual hamiltonian cycle is similar to that of the hypercube. The difference is that once the next node is chosen based on the reflected dimension list in a cluster of a class, we need to go through the cross-edge to a cluster of the other class and do the same work in that cluster. This is the reason why $DD(m)$ doubles each dimension number of $D(m)$. Algorithm 2 shows how to build a hamiltonian cycle in a DC($m$) and hence we have

**Theorem 1.** *There is a hamiltonian cycle in a dual-cube.*

Algorithm 2 (dualcubeHC($m$))
**begin** /* build a hamiltonian cycle $P$ in DC($m$) */
  $DD(m) = \text{DDL}(m)$;
  $EDD(m) = (DD(m), m-1, m-1)$;
  $u = 0$;
  **for** each dimension number $i$ in $EDD(m)$ **do**
    **if** ($u$ is of class 0) $v = u \oplus 2^i$;
    **else** $v = u \oplus 2^{m+i}$;
    $P' = \text{cubeHP}(m, u, v)$;
    $P = P : P'$;
    $u = v \oplus 2^{2m}$;
  **endfor**
**end**
Procedure DDL($m$)
**begin** /* build an double-dimension list for a DC($m$) */
  **if** ($m == 1$) **return** $(0, 0)$;
  **else return** (DDL($m-1$), $m-1$, $m-1$, DDL($m-1$));
**end**

**Lemma 2.** *Given any cube-edge $e = (u : v)$ in a DC(m), there is a virtual hamiltonian cycle going through e.*

***Proof***: Similar to the proof of Lemma 1. □

Since there are $m2^{m-1}$ links in a cluster, by taking each of the links as edge $(u : v)$, we have $m2^{m-1}$ different virtual hamiltonian cycles. These cycles are different but not disjointed.

**Theorem 2.** *There are $2^{m-1}$ disjoint virtual hamiltonian cycles in a DC(m).*

***Proof***: We use induction to prove the theorem. For $m = 2$ (see Figure 4), two links in a cluster, for example $e_0 = (00000 : 00001)$ and $e_1 = (00011 : 00010)$ in the cluster 0 of class 0, connect four distinct nodes in the cluster. Therefore, there are $2^{2-1} = 2$ disjoint virtual hamiltonian cycles that contain $e_0$ and $e_1$, respectively. These two cycles are constructed by $EDD(2)$ based on the reflected dimension list $D(2) = (0, 1, 0)$ with the starting nodes 00000 and 00011, respectively.

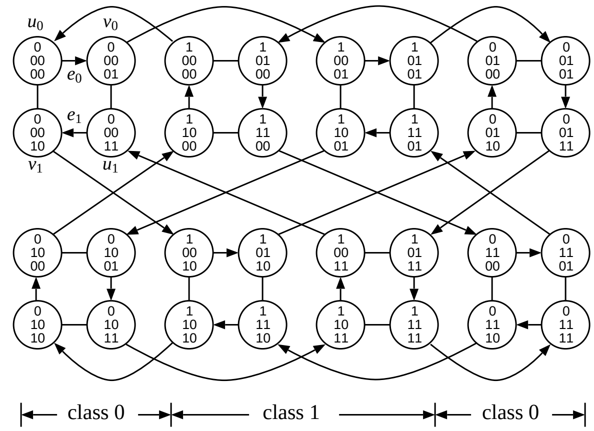



**Figure 4. Disjoint virtual hamiltonian cycles**

Generally, there are $2^m/2$ such links in an $m$-dimensional cluster ($m$-cube): each link takes two nodes from the list of the reflected Gray codes. For $m > 2$, the $2^m/2$ virtual hamiltonian cycles that contain $e_i$, $0 \leq i \leq 2^{m-1} - 1$, respectively, can be built based on the reflected dimension list $D(m)$. Because $D(m) = (D(m-1), m-1, D(m-1))$, by our induction hypothesis, the first half of all the cycles are disjointed. Then, all the paths that go through the $(m-1)$th dimension will still be disjoint. Similarly, the second half of all the cycles are also disjointed. Therefore, all $2^{m-1}$ cycles are disjoint. □

## 4 Fault-Free Cycle Embedding in Dual-Cube with Faulty Nodes

In this section, we consider the problem of finding fault-free cycle of maximal length in dual-cube with faulty nodes. The following lemmas on hypercube are needed.

**Lemma 3.** *Given two links $e_0 = (u_0 : v_0)$ and $e_1 = (u_1 : v_1)$ in an n-cube, there is a hammiltonian cycle which goes through $e_0$ and $e_1$.*

***Proof***: We use induction on $n$ to prove the lemma. For $n = 3$, the lemma is true as shown as in Figure 5. Without loss of generality, let $u_0 = 0$ and $v_0 = 4$. Any of other links, $e_1$, appears in the cycle shown in Figure 5(a), (b) or (c).

We assume that the lemma holds for $n = k \geq 3$. Dividing an $n$-cube along any dimension we can get two $(n-1)$-cubes, namely subcube0 and subcube1, respectively. For $n = k+1$, because there are $n \geq 4$ dimensions, we can devide the $n$-cube along a dimension so that $e_0$ and $e_1$ are in subcube0 and/or subcube1, that is, they do not appear in the dimension with which the $n$-cube was divided.

If $e_0$ and $e_1$ are in a same subcube, subcube0 for instance, by our assumption, there is a hamitonian cycle going through $e_0$ and $e_1$. Select a link $(x : y)$ other than $e_0$ and $e_1$,

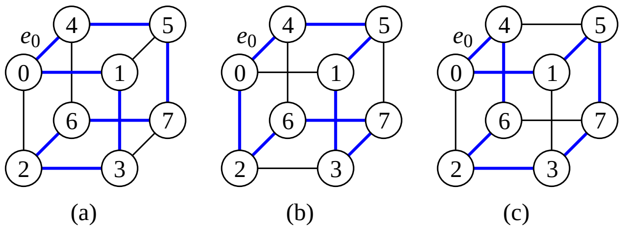



**Figure 5. Links in hamiltonian cycle in 3-cube**

by Lemma 1, there is a hamiltonian cycle going through $(x' : y')$[1] in subcube1. Replacing $(x : y)$ and $(x' : y')$ with $(x : x')$ and $(y : y')$, a hamiltonian cycle going through $e_0$ and $e_1$ is obtained.

If $e_0$ and $e_1$ are in different subcubes, say, $e_0$ is in subcube0 and $e_1$ is in subcube1, by Lemma 1, a hamiltonian cycle going through $e_0$ in subcube0 can be built. Select a link $(x : y)$ so that $(x : y) \neq e_0$ and $(x' : y') \neq e_1$, by our assumption, there is a hamitonian cycle going through $(x', y')$ and $e_1$ in subcube1. Replacing $(x : y)$ and $(x' : y')$ with $(x : x')$ and $(y : y')$, a hamiltonian cycle going through $e_0$ and $e_1$ is obtained. □

**Lemma 4.** *Given three links $e_0 = (u_0 : v_0)$, $e_1^0 = (u_1 : w)$ and $e_1^1 = (w : v_1)$ in an n-cube, where $w \neq u_0$ and $w \neq v_0$, there is a hammiltonian cycle which goes through $e_0$, $e_1^0$ and $e_1^1$.*

***Proof***: We use induction on $n$ to prove the lemma. For $n = 3$, the lemma is true as shown as in Figure 5. Without loss of generality, let $u_0 = 0$ and $v_0 = 4$. For $w = 1$, all three link patterns are shown in Figure 5(a), (b) and (c), respectively. The cases of $w = 2,\ 5,\ 6$ are similar to the case of $w = 1$. For $w = 3$, all three link patterns are shown in the figure, and the cases of $w = 7$ are similar to the case of $w = 3$.

We assume that the lemma holds for $n = k \geq 3$. For $n = k+1$, because there are $n \geq 4$ dimensions, we can devide the $n$-cube along a dimension so that $e_0 = (u_0 : v_0)$, $e_1^0 = (u_1 : w)$ and $e_1^1 = (w : v_1)$ do not appear in the dimension with which the $n$-cube was divided. Note that $e_1^0 = (u_1 : w)$ and $e_1^1 = (w : v_1)$ are in a same subcube. Assume that $e_0$ is in subcube0.

If $e_1^0 = (u_1 : w)$ and $e_1^1 = (w : v_1)$ are in subcube0, by our assumption, there is a hamitonian cycle going through $e_0 = (u_0 : v_0)$, $e_1^0 = (u_1 : w)$ and $e_1^1 = (w : v_1)$. Select a link $(x : y)$ other than $e_0 = (u_0 : v_0)$, $e_1^0 = (u_1 : w)$ and $e_1^1 = (w : v_1)$, by Lemma 1, there is a hamiltonian cycle going through $(x' : y')$ in subcube1. Replacing $(x : y)$ and $(x' : y')$ with $(x : x')$ and $(y : y')$, a hamiltonian cycle going through $e_0$ and $e_1$ is obtained.

[1]The addresses of $x$ and $x'$ diff only in a bit position – the dimension with which the $n$-cube was divided, so as $y$ and $y'$.

If $e_1^0 = (u_1 : w)$ and $e_1^1 = (w : v_1)$ are in subcube1. By Lemma 1, a hamiltonian cycle going through $e_0$ in subcube0 can be built. Select a link $(x : y)$ other than $e_0$ so that $(x' : y') \neq e_1^0$ and $(x' : y') \neq e_1^1$, by our assumption, there is a hamitonian cycle going through $(x', y')$, $e_1^0 = (u_1 : w)$ and $e_1^1 = (w : v_1)$ in subcube1. Replacing $(x : y)$ and $(x' : y')$ with $(x : x')$ and $(y : y')$, a hamiltonian cycle going through $e_0$ and $e_1$ is obtained. □

**Lemma 5.** *Given a link $e = (u : v)$ in an n-cube with $f \leq n-2$ faulty nodes, where u and v are two non-faulty nodes connected by a link e, there is a fault-free cycle which contains at least $2^n - 2f$ nodes and goes through link e.*

***Proof***: We use induction on $n$ to prove the lemma. For $n = 3$, the lemma is true as shown as in Figure 6, where $u = 3$ and $v = 7$. The figure shows the case of node 0 faulty. The case of node 4 faulty is similar. If node 1 is fault, the cycle is the same as in the figure and the case in which node 2, 5, or 6 is fault is similar to the case of node 1 faulty.

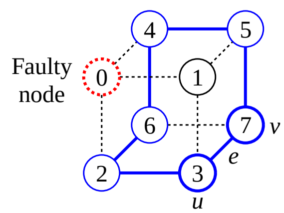



**Figure 6. Fault-free cycle in 3-cube**

We assume that the lemma holds for $n = k \geq 3$. For $n = k+1$, without loss of generality, assume that $e$ belongs to subcube0. Let $f_0$ and $f_1$ be the numbers of faulty nodes in subcube0 and subcube1, respectively, where $f_0 + f_1 = f \leq k-1$. The proof of the lemma is divided into three cases.

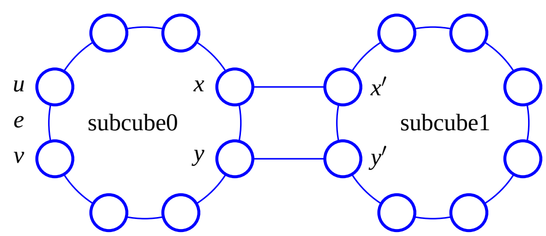



**Figure 7. Fault-free cycle in *n*-cube (case 1)**

**Case 1**: $f_0 < f$ and $f_1 < f$. By our assumption, there is a fault-free cycle containing at least $2^k - 2f_0$ nodes going through link $e$ in subcube0. Let $(x : y)$ be a link in the fault-free cycle so that the corresponding nodes $x'$ and $y'$ in subcube1 are not fault. By our assumption, in subcube1, there is a fault-free cycle which contains at least $2^k - 2f_1$ nodes and goes through link $(x' : y')$. By replacing the links

$(x:y)$ and $(x':y')$ with the links $(x:x')$ and $(y:y')$, respectively, a fault-free cycle can be built that contains at least $(2^k-2f_0)+(2^k-2f_1)=2^{k+1}-2f$ nodes and goes through link $e$ in $(k+1)$-cube.

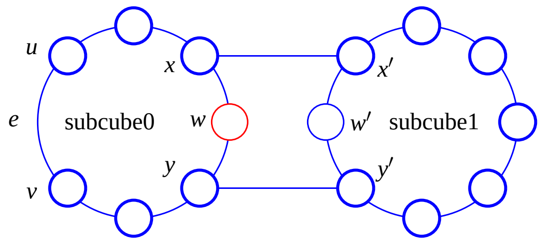



**Figure 8. Fault-free cycle in *n*-cube (case 2)**

**Case 2**: $f_0 = f$. Let $w$ be a faulty node. Suppose $w$ appears in the hamiltonian cycle built in Case 1. Let $x$ and $y$ be the two neighbors of $w$ in the cycle. Because no faulty node exists in subcube1, there is a hamiltonian cycle of length $2^k$ such that the $x'$ and $y'$ are the two neighbors of $w'$ in the cycle. By replacing the links $(x':w')$ and $(w':y')$ with the links $(x:x')$ and $(y:y')$, respectively, a fault-free cycle can be built that contains at least $(2^k-2(f-1)-1)+(2^k-1)=2^{k+1}-2f$ nodes and goes through link $e$ in $(k+1)$-cube.

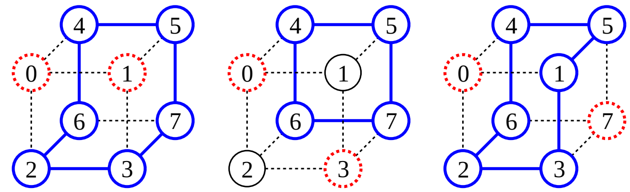
**Case 3**: $f_1 = f$. Note that we can select a dimension to divide the $n$-cube so that $u'$ and $v'$ are not fault. Because, if $u'$ and $v'$ are fault, we can divide the $n$-cube so that at least one of $u'$ and $v'$ is in subcube0 and apply the proof of Case 1 or Case 2.

Let $w'$ be a faulty node. Suppose $w'$ appears in the hamiltonian cycle built in Case 1. Let $x'$ and $y'$ be the two neighbors of $w'$ in the cycle. Because no faulty node exists in subcube0, by applying Lemma 4, there is a hamiltonian cycle of length $2^k$ such that the $x$ and $y$ are the two neighbors of $w$ in the cycle. By replacing the links $(x:w)$ and $(w:y)$ with the links $(x:x')$ and $(y:y')$, respectively, a fault-free cycle can be built that contains at least $(2^k-1)+(2^k-2(f-1)-1)=2^{k+1}-2f$ nodes and goes through link $e$ in $(k+1)$-cube. $\square$

**Lemma 6.** *There is a fault-free cycle containing at least* $2^n-2f$ *nodes in an n-cube with* $f \le n-1$ *faulty nodes.*

***Proof***: We use induction on $n$ to prove the lemma. The lemma is true for $n=3$ as shown as in Figure 9, where two faulty nodes are denoted by dotted cycles. We assume that the lemma holds for $n=k\ge 3$. For $n=k+1$, let $f_0$ and $f_1$ be the numbers of faulty nodes in subcube0 and subcube1, respectively, where $f_0+f_1=f\le k$. For $f=k\ge 3$, we can always select a dimension to divide the $n$-cube so that $f_0<f$ and $f_1<f$.

By our assumption, there is a fault-free cycle containing at least $2^k-2f_0$ nodes in subcube0. It is true that either




(a) $d=1$ (b) $d=2$ (c) $d=3$

**Figure 9. Fault-free cycle in 3-cube**

$f_0\le k-2$ or $f_1\le k-2$: Because, otherwise, $f=f_0+f_1\ge 2k-2>k$, for $k\ge 3$, but our assumption is $f=f_0+f_1\le k$. Without loss of generality, assume $f_1\le k-2$. Let $(x:y)$ be a link in the fault-free cycle in subcube0 so that the corresponding nodes $x'$ and $y'$ in subcube1 are not fault. By applying Lemma 5, a fault-free cycle containing at least $(2^k-2f_0)+(2^k-2f_1)=2^{k+1}-2f$ nodes and goes through link $e$ in $(k+1)$-cube. $\square$

Now, we show that there is a fault-free cycle containing at least $2^{2m+1}-2f$ nodes with $f\le m-1$ faulty nodes in a DC($m$). By Theorem 2, there are $2^{m-1}$ disjoint virtual hamiltonian cycles in a DC($m$). Because $f\le m-1<2^{m-1}$, there exists a virtual hamiltonian cycle that contains no faulty node. Let $f_i$, $i=1,2,\ldots,h$ be the numbers of faulty nodes in the $h$ distinct clusters ($m$-cubes), respectively, where $\sum_{i=1}^{h} f_i = f \le m-1$.

We first consider that $f_i<f$ for $i=1,2,\ldots,h$. By Lemma 5, there exists a fault-free cycle containing at least $2^m-2f_i$ nodes in each of $h$ $m$-cubes for $i=1,2,\ldots,h$, that goes through the link in the virtual hamiltonian cycle. Therefore, it is easy to construct a fault-free cycle in the DC($m$) that contains $\sum_{i=1}^{h}(2^m-2f_i)+(2^{m+1}-h)2^m = 2^{2m+1}-2f$ nodes.

Then we consider all the faulty nodes appear in a same cluster. Without loss of generality, assume that cluster 0 contains all the $f\le m-1$ faulty nodes. By Lemma 6, there exists a fault-free cycle containing at least $2^m-2f$ nodes in cluster 0. Let $e_0$ be a link in this fault-free cycle. By Lemma 2, we can construct a virtual hamiltonian cycle in DC($m$) that contains $e_0$. Then we replace $e_0$ with the path of length at least $2^m-2f-1$ in cluster 0, and in each of the other $2^{m+1}-1$ clusters, we replace each link in the virtual hamiltonian cycle with a hamiltonian path. Thus, a fault-free cycle in DC($m$) containing at least $(2^m-2f)+(2^{m+1}-1)2^m=2^{2m+1}-2f$ nodes can be built. We summarize these results in the following theorem.

**Theorem 3.** *There is a fault-free cycle containing at least* $2^{2m+1}-2f$ *nodes in a DC(m) with f faulty nodes, where* $f\le m-1$.

**Table 1. Properties of fault tolerance: dual-cube *vs* hypercube**

| | Number of nodes | Link faulty | Node faulty |
|---|---|---|---|
| Hypercube | $2^n$ | $(n-2)$-hamiltonian | Faulty-free cycle contains $2^n - 2f$ nodes, $f \leq n-1$ |
| Dual-cube | $2^{2m+1}$ | $(m-1)$-hamiltonian | Faulty-free cycle contains $2^{2m+1} - 2f$ nodes, $f \leq m-1$ |

Table 1 summarize the properties of fault tolerance of the dual-cube and hypercube where $n = m+1$. The dual-cube holds almost the same properties as the hypercube but can connect much more nodes than the hypercube with the same number of links per node.

## 5 Conclusion and Future Work

In this paper, we showed that a fault-free cycle containing at least $2^{2m+1} - 2f$ nodes can be constructed in a DC($m$) with $f \leq m-1$ faulty nodes. Because a dual-cube can link much more nodes than other variations of hypercube, it could be used as an interconnection network for large scale parallel computers.

Recently, much of the community has moved on to lower-dimensional topologies such as meshes and tori. However, the SGI Origin2000, a fairly recent multiprocessor, does use a hypercube topology, so the dual-cube could be of use to industry. A lot of issues concerning the dual-cube require further research. Some of them are:

1. Evaluate the architecture complexity vs. performance of benchmarks vs. real cost.
2. Investigate the embedding of other frequently used topologies into a dual-cube.
3. Develop techniques for mapping application algorithms onto a dual-cube.
4. Develop fault-tolerant routing algorithms for a dual-cube with faulty nodes.

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